U-grad or M-Eng or PhD? & 4730 or 5730?

"SOLUTIONS"

Your name:

ANDY RUINA

Cornell
ME 4730/5730

Prelim 2

Nov 18, 2014

No calculators, books or notes allowed.

3 Problems, 90 minutes (+ up to 90 minutes extra if you want it)

How to get the highest score?

Please do these things:

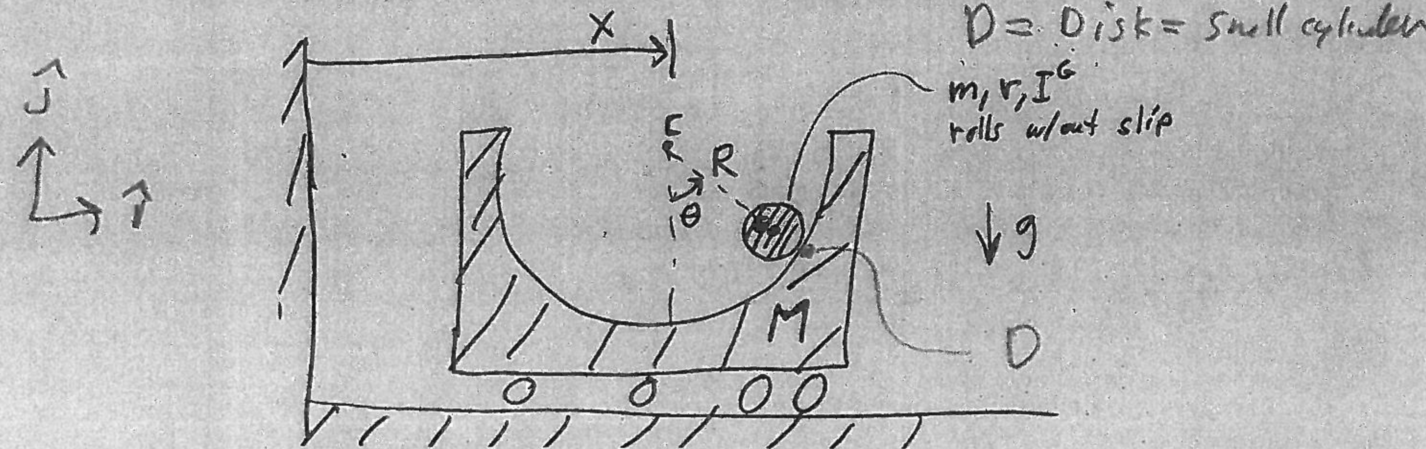
- Draw **Free body diagrams** whenever force, moment, linear momentum, or angular momentum balance are used.
- Use correct **vector notation**.
- A+ Be (I) neat, (II) clear and (III) well organized.
- TIDILY REDUCE and $\boxed{\text{box in}}$ your answers (Don't leave simplifyable algebraic expressions).
- >> Make appropriate `Matlab code` clear and correct. You can use shortcut notation like "$T_7 = 2\pi$" instead of, say, "`T(7) = 2*pi;`". Small syntax errors will have small penalties.
- Clearly **define** any needed dimensions ($\ell, h, d, \ldots$), coordinates ($x, y, r, \theta \ldots$), variables ($v, m, t, \ldots$), base vectors ($\hat{\imath}, \hat{\jmath}, \hat{e}_r, \hat{e}_\theta, \hat{\lambda}, \hat{n} \ldots$) and signs ($\pm$) with sketches, equations or words.
- **Justify** your results so a grader can distinguish an informed answer from a guess.
- If a problem seems ***poorly defined***, clearly state any reasonable assumptions (that do not oversimplify the problem).
- $\approx$ Work for **partial credit** (from 60–100%, depending on the problem)
  - Put your answer is in terms of well defined variables even if you have not substituted in the numerical values.
  - Reduce the problem to a clearly defined set of equations to solve.
  - Provide Matlab code which would generate the desired answer (and explain the nature of the output).
- **Extra sheets.** The last page is blank for your use. Ask for more extra paper if you need it. Put your name on each extra sheet, fold it in, and refer to back pages or extra sheets on the page of the relevant problem.

Problem 4: ______ /25

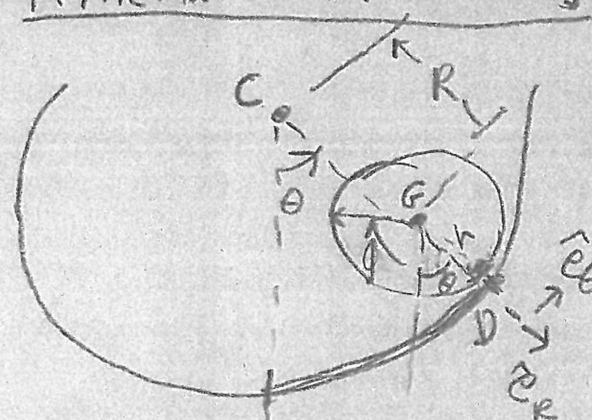
Problem 5: ______ /25

Problem 6: ______ /25

1) Find, by whatever means pleases you (but not allowing computer algebra), the equations of motion of the system below. You need not invert the mass matrix (that is, you need not solve for $\ddot{\theta}$ and $\ddot{x}$). That is, write a clear set of equations from which you could find $\ddot{\theta}$ and $\ddot{x}$ if given $m, M, r, R, I^G, g, \theta, \dot{\theta}, x$ and $\dot{x}$. R is the distance from the center of the hollow cylinder to the center of the rolling cylinder.



Kinematics of Rolling



Lots of ways to find rel. bet θ & ϕ.
Here's one.

arc length = arc length

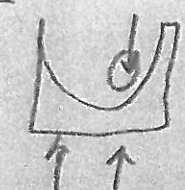
$$(R+r)\,\theta = r(\theta+\phi)$$

$$R\theta = r\phi$$

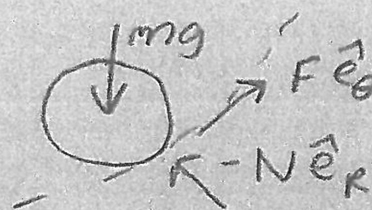
$$\Rightarrow \vec{\omega}_d = -\dot{\phi}\hat{k} = \frac{-R}{r}\dot{\theta}\hat{k}$$

FBDs:

System



disk



$$\hat{e}_R \cdot \hat{\imath} = \sin\theta$$
$$\hat{e}_\theta \cdot \hat{\imath} = \cos\theta$$

{System LMB} $\cdot\,\hat{\imath} \Rightarrow \{\sum \vec{F}\}\cdot\hat{\imath} = \{\sum m_i \vec{a}_i\}\cdot\hat{\imath}$

$$\Rightarrow \vec{0} = M\ddot{x} + m\Big(\underbrace{\ddot{x}\hat{\imath}}_{\vec{a}_C} + \underbrace{-R\dot{\theta}^2\hat{e}_R + R\ddot{\theta}\hat{e}_\theta}_{\vec{a}_{G/C}}\Big)\cdot\hat{\imath}$$

$$\Rightarrow \boxed{0 = (M+m)\ddot{x} + -mR\dot{\theta}^2\sin\theta + R\ddot{\theta}\cos\theta\, m} \quad (1)$$

Cylinder AMB/D

$$\sum \vec{M}_{/D} = \vec{r}_{G/D} \times m\vec{a}_G + I^G(\ddot{\phi}\hat{k})$$

$$\underbrace{\vec{r}_{G/D}}_{-r\hat{e}_R} \times (-mg\hat{j}) = (-r\vec{e}_R) \times [\underbrace{\vec{a}_C}_{\ddot{x}\hat{i}} + \underbrace{\vec{a}_{G/C}}_{R\ddot{\theta}\vec{e}_\theta - R\dot{\theta}^2\hat{e}_R}] + I^G \underbrace{\ddot{\phi}}_{-\frac{R}{r}\ddot{\theta}} \hat{k}$$

$\omega_d\hat{k}$

$$\Rightarrow \{+mgr\sin\theta\,\hat{k} = [-r\cos\theta\,\ddot{x}\hat{k} - rR\ddot{\theta}\hat{k}]m - I^G\frac{R}{r}\ddot{\theta}\hat{k}\}$$

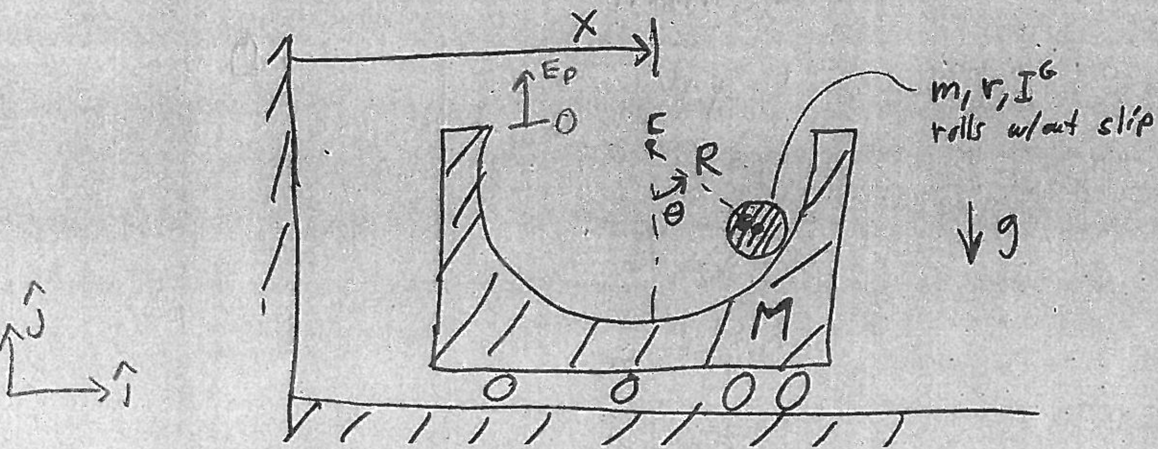
$$-\{\}\cdot\hat{k} \Rightarrow \boxed{-mgr\sin\theta = [+r\cos\theta\,\ddot{x} + rR\ddot{\theta}]m + I_G\frac{R}{r}\ddot{\theta}} \quad (2)$$

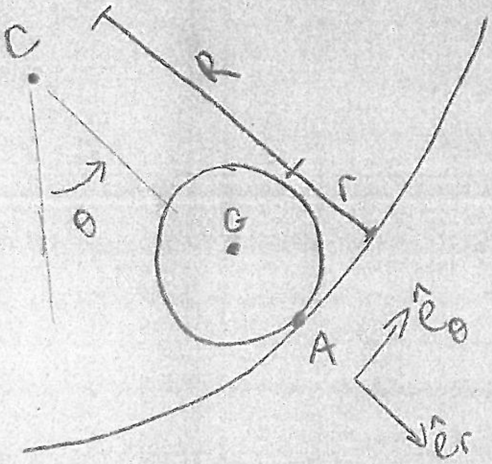
(1) & (2)

$$\boxed{\underbrace{\begin{bmatrix} M+m & Rm\cos\theta \\ mr\cos\theta & rRm + I_G\frac{R}{r} \end{bmatrix}}_{M} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} mR\dot{\theta}^2\sin\theta \\ -mgr\sin\theta \end{bmatrix}} \quad (3)$$

Ans: (1) & (2) or (3)

1) Find, by whatever means pleases you (but not allowing computer algebra), the equations of motion of the system below. You need not invert the mass matrix (that is, you need not solve for $\ddot{\theta}$ and $\ddot{x}$). That is, write a clear set of equations from which you could find $\ddot{\theta}$ and $\ddot{x}$ if given $m, M, r, R, I^G, g, \theta, \dot{\theta}, x$ and $\dot{x}$. R is the distance from the center of the hollow cylinder to the center of the rolling cylinder.





Kinematics:

$$\vec{V}_{G/C} = \underbrace{\vec{V}_{A/C}}_{\vec{0}} + \vec{V}_{G/A}$$

$$\vec{r}_{G/C} = R\hat{e}_r \rightarrow \vec{V}_{G/C} = R\dot{\theta}\hat{e}_\theta$$

$$\vec{r}_{G/A} = -r\hat{e}_r \rightarrow \vec{V}_{G/A} = -\omega r\hat{e}_\theta$$

$$R\dot{\theta} = -\omega r$$

$$\hat{e}_r = \sin\theta\,\hat{\imath} - \cos\theta\,\hat{\jmath}$$

$$\hat{e}_\theta = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$$

$\boxed{\text{Lagrange}}$

Hollow cylinder: $E_K = \frac{1}{2}M\dot{x}^2$

Rolling cylinder: $\vec{r}_G = x\hat{\imath} + R\hat{e}_r \rightarrow \vec{V}_G = \dot{x}\hat{\imath} + R\dot{\theta}\hat{e}_\theta$

$$\vec{V}_G = (\dot{x} + R\dot{\theta}\cos\theta)\hat{\imath} + R\dot{\theta}\sin\theta\,\hat{\jmath}$$

$$\vec{V}_G \cdot \vec{V}_G = \dot{x}^2 + 2\dot{x}R\dot{\theta}\cos\theta + R^2\dot{\theta}^2\cos^2\theta + R^2\dot{\theta}^2\sin^2\theta$$

$$= \dot{x}^2 + 2\dot{x}R\dot{\theta}\cos\theta + R^2\dot{\theta}^2$$

$$E_K = \tfrac{1}{2} m \left(\dot{x}^2 + 2\dot{x} R \dot{\theta} \cos\theta + R^2 \dot{\theta}^2\right) + \tfrac{1}{2} I^G \omega^2$$ 3

$$E_p = - mgR\cos\theta$$

$$\mathcal{L} = \tfrac{1}{2}(M+m)\dot{x}^2 + m\dot{x}R\dot{\theta}\cos\theta + \tfrac{1}{2} m R^2 \dot{\theta}^2 + \tfrac{1}{2} I^G \left(\frac{R^2}{r^2}\right)\dot{\theta}^2$$
$$+ mgR\cos\theta$$

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \qquad \frac{\partial \mathcal{L}}{\partial \dot{x}} = (M+m)\dot{x} + mR\dot{\theta}\cos\theta$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = (M+m)\ddot{x} + mR\ddot{\theta}\cos\theta - mR\dot{\theta}^2 \sin\theta$$

$$\boxed{0 = (M+m)\ddot{x} + mR\ddot{\theta}\cos\theta - mR\dot{\theta}^2\sin\theta} \qquad 1$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -m\dot{x}R\dot{\theta}\sin\theta - mgR\sin\theta$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m\dot{x}R\cos\theta + mR^2\dot{\theta} + I^G\left(\frac{R}{r}\right)^2\dot{\theta}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m\ddot{x}R\cos\theta - m\dot{x}R\dot{\theta}\sin\theta + mR^2\ddot{\theta} + I^G\left(\frac{R}{r}\right)^2\ddot{\theta}$$

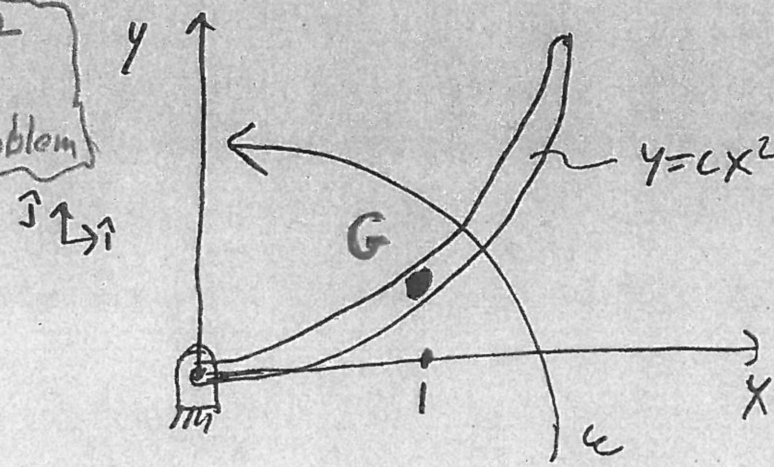
$$-m\dot{x}\cancel{R\dot{\theta}\sin\theta} - mgR\sin\theta = m\ddot{x}R\cos\theta - \cancel{m\dot{x}R\dot{\theta}\sin\theta}$$
$$+ mR^2\ddot{\theta} + I^G\left(\frac{R}{r}\right)^2\ddot{\theta}$$

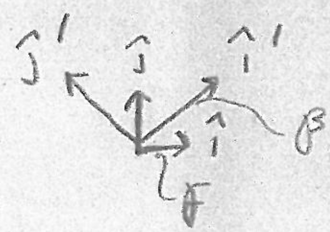
$$\boxed{-mgR\sin\theta = mR\ddot{x}\cos\theta + \ddot{\theta}\left(mR^2 + I^G\left(\frac{R}{r}\right)^2\right)}$$

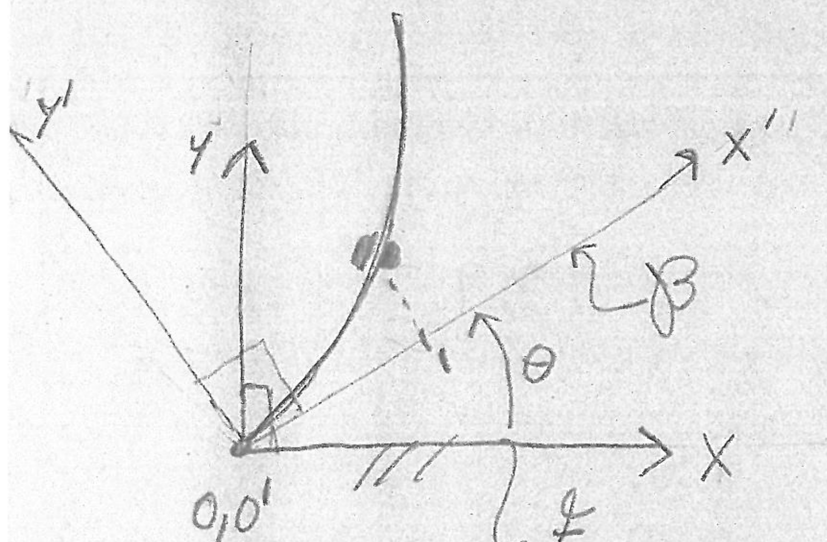
$$= 2 \cdot \left(\frac{R}{r}\right)$$

2) A rigid tube has the shape $y' = cx'^2$. It is rotated about the origin at constant ω along with it's $x' - y'$ coordinate system. Inside the tube a particle moves, due to magical forces, with constant $\dot{x}' = v_o$. At the instant of interest the $x' - y'$ axis is coincident with the $x - y$ axis and the bead is at $x' = 1$ (in some consistent units). Find the acceleration $\vec{a}$ of the bead in terms of $v_0, c, \omega, \hat{\imath}$ and $\hat{\jmath}$.

Motion is given
⇒ this is a kinematics problem







$$\vec{\omega}_{\mathcal{B}/\mathcal{F}} = \omega\hat{k} = \text{const}$$

$$\dot{x}' = V_0 \Rightarrow \ddot{x}' = 0$$

$$\dot{y}' = \frac{d}{dt}(cx'^2) = 2cx'\dot{x}' = 2cx'V_0$$

$$\ddot{y}' = 2c\dot{x}'V_0 = 2cV_0^2$$

$$\boxed{\vec{a}_{G/\mathcal{B}} = 2cV_0^2\hat{\jmath}'}$$

$$\vec{a}_G = \overset{\vec{0}}{\cancel{\vec{a}_{O'}}} + \vec{a}_{G/\mathcal{B}} + -\omega^2\vec{r}_{G/O'} + \overset{\vec{0}}{\cancel{\dot{\vec{\omega}}_{\mathcal{B}/\mathcal{F}}}}\times\vec{r}_{G/O'} + 2\underbrace{\vec{\omega}_{\mathcal{B}/\mathcal{F}}}_{\omega\hat{k}}\times\underbrace{\vec{V}_{G/\mathcal{B}}}_{V_0\hat{\imath}' + 2cV_0\hat{\jmath}'} \qquad 1$$

$$\vec{r}_{G/O'} = x'\hat{\imath}' + y'\hat{\jmath}'$$

At instant of interest:

$\hat{i} = \hat{i}'$ $\quad x' = x = 1$ $\quad \vec{r}_{G/O} = \hat{i} + c\hat{j}$

$\hat{j} = \hat{j}'$ $\quad y' = y = c$

(1) $\Rightarrow$

$$\vec{a}_G = \vec{0} + 2cV_0^2\hat{j} - \omega^2(\hat{i} + c\hat{j}) + 2\omega\hat{k} \times [V_0\hat{i}' + 2cV_0\hat{j}']$$

$$= 2cV_0^2\hat{j} - \omega^2\hat{i} - \omega^2 c\hat{j} + 2\omega V_0\hat{j} - 4c\omega V_0\hat{i}$$

$$\boxed{\vec{a}_G = [-\omega^2 - 4c\omega V_0]\,\hat{i} + [2cV_0^2 - \omega^2 c + 2\omega V_0]\,\hat{j}}$$

Note: can't do length units check because lengths given unitless

Checks: * $\omega = 0 \Rightarrow \vec{a}_G = 2cV_0^2\hat{j}$ ✓

* $V_0 = 0 \Rightarrow \vec{a}_G = -\omega^2\hat{i} - \omega^2 c\hat{j}$

$$= -\omega^2(\underbrace{\hat{i} + c\hat{j}}_{\vec{r}_{G/O}})$$ ✓

* time units check: all terms $(1/t)^2$ ✓

3) Consider the equation for the undamped sinusoidally forced harmonic oscillator:

$$m\ddot{x} + kx = F_o \sin(\omega t)$$

a) Assuming $\omega \neq \sqrt{k/m}$ find a particular solution $x(t)$ to the governing equation.

b) As accurately as you can, plot the amplitude and phase of that solution (noting any key points on the axes).

(As a function of ω)

c) Assume $x(0) = 1$ and $\dot{x}(0) = 0$, find $x(t)$.

a) Guess $x_p = A\sin(\lambda t + \phi)$

$\dot{x}_p = A\lambda\cos(\lambda t + \phi)$

$\ddot{x}_p = -A\lambda^2 \sin(\lambda t + \phi)$

$-A\lambda^2 m \sin(\lambda t + \phi) + Ak\sin(\lambda t + \phi) = F_o \sin(\omega t)$

$\lambda = \omega \qquad \phi = 0$

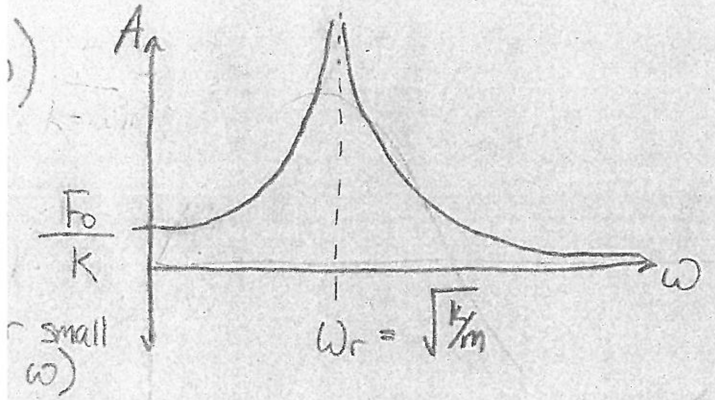
$-A\omega^2 m + Ak = F_o$

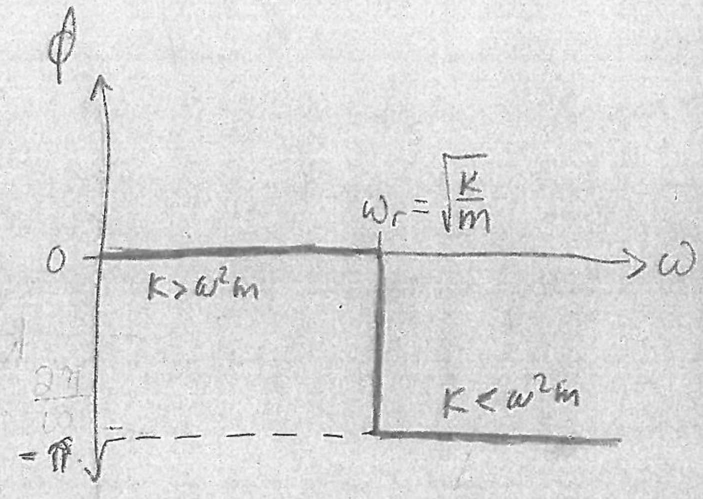
$A(k - \omega^2 m) = F_o$

$A = \dfrac{F_o}{k - \omega^2 m}$

$x_p = \underbrace{\left[\dfrac{F_o}{k-\omega^2 m}\right]}_{A} \sin(\omega t)$

b)





c) Guess $x_h = B\sin\left(\sqrt{\frac{k}{m}}t\right) + C\cos\left(\sqrt{\frac{k}{m}}t\right)$

$$\dot{x}_h = B\sqrt{\tfrac{k}{m}}\cos\left(\sqrt{\tfrac{k}{m}}t\right) - C\sqrt{\tfrac{k}{m}}\sin\left(\sqrt{\tfrac{k}{m}}t\right)$$

$$\ddot{x}_h = -B\frac{k}{m}\sin\left(\sqrt{\tfrac{k}{m}}t\right) - C\frac{k}{m}\cos\left(\sqrt{\tfrac{k}{m}}t\right)$$

$$-mB\frac{k}{m}\sin\left(\sqrt{\tfrac{k}{m}}t\right) - mC\frac{k}{m}\cos\left(\sqrt{\tfrac{k}{m}}t\right) + kB\sin\left(\sqrt{\tfrac{k}{m}}t\right) + kC\cos\left(\sqrt{\tfrac{k}{m}}t\right) = 0$$

General solution:

$$x(t) = B\sin\left(\sqrt{\tfrac{k}{m}}t\right) + C\cos\left(\sqrt{\tfrac{k}{m}}t\right) + \frac{F_0}{k-\omega^2 m}\sin(\omega t)$$

$$x(0) = 1 = 0 + C + 0$$

$$C = 1$$

$$\dot{x}(t) = B\sqrt{\tfrac{k}{m}}\cos\left(\sqrt{\tfrac{k}{m}}t\right) - C\sqrt{\tfrac{k}{m}}\sin\left(\sqrt{\tfrac{k}{m}}t\right) + \frac{F_0\omega}{k-\omega^2 m}\cos(\omega t)$$

$$\dot{x}(0) = 0 = B\sqrt{\tfrac{k}{m}} + \frac{F_0\omega}{k-\omega^2 m}$$

$$B = -\sqrt{\frac{m}{k}}\frac{F_0\omega}{k-\omega^2 m}$$

$$\boxed{x(t) = -\left(\sqrt{\frac{m}{k}}\frac{F_0\omega}{k-\omega^2 m}\right)\sin\left(\sqrt{\tfrac{k}{m}}t\right) + \cos\left(\sqrt{\tfrac{k}{m}}t\right) + \frac{F_0}{k-\omega^2 m}\sin(\omega t)}$$